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THE CLOSE-PACKED-SPHERON MODEL OF ATOMIC NUCLEI AND ITS RELATION TO THE SHELL MODEL

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I have found that the assumption that in atomic nuclei the nucleons are in large part aggregated into clusters arranged in closest packing leads to simple explanations of many properties of nuclei. Some aspects of the closest-packing theory of nuclear structure are presented in the following paragraphs.[1]

The treatment of nuclei such as O^{16} as aggregates of helions (alpha particles[2]) has been reasonably successful.[3] The binding energy of nucleons in O^{16} is only 13 per cent greater than that of four helions, and accordingly a wave function composed of helion wave functions would be expected to be a good zero-order approximation. For example, four equivalent orbitals centered about the corners of a tetrahedron can be formed by sp^3 hybridization of the orbitals of the shell model,[4–6] which are referred to the center of the O^{16} nucleus. This process of hybridization of orbitals provides the formal basis of the relation between the helion model (cluster model) and the shell model. Nuclei with more neutrons than protons may be described in terms of helions, tritons ($H^3 = n^2p$), and dineutrons (plus $He^3 = np^2$ for N odd). These clusters of neutrons and protons occupying localized $1s$ orbitals may be called *spherons:* the largest spheron is the helion.

There is evidence from electron-scattering experiments[7] that the density of protons is approximately constant throughout a spherical central region of a nucleus, and then falls off to zero with increasing radius, in a way that has been described[7] as constituting a "skin" with thickness about $2.4f$. This skin thickness may be ascribed in part to the nubbling of the surface by the outer hemispheres of the helions and tritons of the outermost layer of the nucleus. The observed charge distribution of the helion corresponds to a radius of about $2f$, and the packing volume $24f^3$ to the radius $1.62f$ ($1.47f$ for the triton).

The Distribution of Spherons in Layers.—Several theoretical and empirical arguments indicate that the nature of spheron-spheron interactions is not such as to limit the ligancy of a spheron to a fixed value, but that, instead, maximum stability is achieved when each spheron ligates about itself the maximum number of neighbors; aggregates of spherons, like aggregates of argonon (noble-gas) atoms or metal atoms, assume a closest-packed structure.

A simple argument leads us to conclude that the spherons in a nucleus are arranged approximately in a series of concentric layers. The energy of a nucleus is minimal for spherical (or ellipsoidal) shape; deviation from this shape, with surface irregularities greater than the spheron nubbling, decreases the number of spheron-spheron interactions and thus increases the surface energy. We conclude that there is a surface layer of spherons. For a large nucleus, the outer part of the cavity inside this surface layer is occupied by spherons in contact with the inner side of the surface layer. These spherons hence constitute another layer, within which there may be still another spheron or layer of spherons.

To avoid confusion with the shells of the shell model of the nucleus we shall refer to the layers of spherons by special names: the *mantle* for the surface layer, and the *outer core* and *inner core* for the two other layers of a three-layer nucleus.

For soft spheres, permitting about 10 per cent variation in contact distance, triangular packing (tetrahedral packing) is denser than ordinary "closest" packing, as represented in metals such as copper and magnesium. In triangular packing the coordination polyhedra have triangular faces and each sphere in an outer layer lies out from the center of a triangle, thus forming a tetrahedron. Pure triangular packing is limited to finite aggregates. The icosahedral packing shown in Figure 1, involving three layers of spheres surrounding a central sphere, is found[8] around each lattice point of the body-centered cubic lattice of the intermetallic compound $Mg_{32}(Al,Zn)_{49}$.

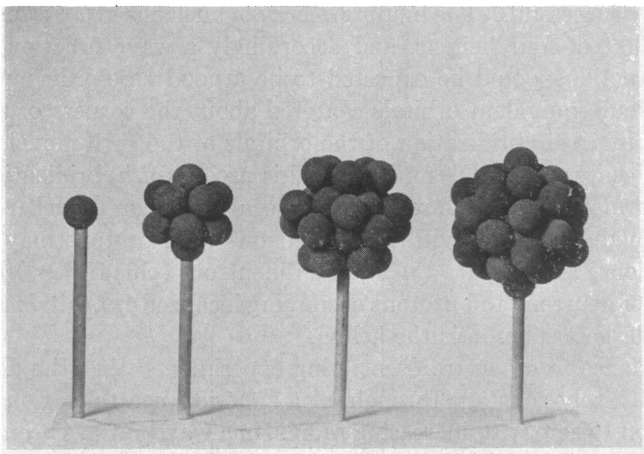

FIG. 1.—The arrangement of 45 spheres in icosahedral closest packing. At the left there is shown a single sphere, which constitutes the inner core. Next there is shown the layer of 12 spheres, at the corners of a regular icosahedron. The third model shows the core of 13 spheres with 20 added in the outer layer, each in a triangular pocket corresponding to a face of the icosahedron; these 20 spheres lie at the corners of a pentagonal dodecahedron. The third layer is completed, as shown in the model at the right, by adding 12 spheres at corners of a large icosahedron; the 32 spheres of the third layer lie at the corners of a rhombic triacontahedron. The fourth layer (not shown) contains 72 spheres.

We may use this example of triangular closest packing to derive an expression for the distribution of spheres in successive layers. The form of the expression (number of spheres proportional to the cube of a length, the radius) reflects the assumption of constancy of effective volume per sphere. The expression is

$$n_t = (n_i^{1/3} + 1.30)^3, \qquad (1)$$

which may be rewritten as

$$n_i = (n_t^{1/3} - 1.30)^3. \quad (1a)$$

Here n_t is the total number of spheres, and n_i the number not including those in the outermost layer. The number 1.30, representing the effective thickness of the outer layer, is obtained in the following way. For the structure shown in Figure 1 the numbers of spheres in successive layers, from the center out, are 1, 12, 32, 72, and the successive total numbers are 1, 13, 45, 117. The value 1.30 for the constant in equation (1) provides the best simple approximation to these numbers: it gives 0.97, 12.0, 46.0, 117. The amount of agreement suggests that equation (1) is reliable to about ± 1, and the following discussion corresponds to this assumption.

The Relation between the Shell Model and Layers of Spherons.—In the customary nomenclature for nucleon orbitals the principal quantum number n is taken to be $n_r + 1$, where n_r, the radial quantum number, is the number of nodes in the radial wave function. (For electrons n is taken to be $n_r + l + 1$.) The nucleon distribution function for $n = 1$ corresponds to a single shell (for $1s$ a ball) about the origin. For $n = 2$ the wave function has a small negative value inside the nodal surface, that is, in the region where the wave function for $n = 1$ and the same value of l is large, and a large value in the region just beyond this surface.

The nature of the radial wave functions thus leads us to the following interpretation[1] of the subshells of the shell model:

1. Those subshells that occur (are occupied) with only the value 1 for the quantum number n contribute only to the outer layer of spherons (the mantle).
2. Those subshells that occur with two values of n contribute to the mantle and the next inner layer, and so on.

Thus for the neutron configuration $1s^2 2s^2 1p^6 1d^{10} 3s^2 2p^6 2d^{10} 1f^{14} 1g^{18} (1h\ 11/2)^{12}$ we assign $1s^2$ to the inner core (2 neutrons), $2s^2 1p^6 1d^{10}$ to the outer core (18 neutrons), and the remainder (62 neutrons) to the mantle.

The Assignment of Nucleons to Layers by Use of the Packing Equation.—Equation (1) has been applied in the assignment of neutrons to the mantle, outer core, inner core, and innermost core (for N very large), with the results shown in Figure 2.

The method may be illustrated by the example $N = 20$, as in Ca^{40}, to which we assign ten spherons (ten helions). For $n_t = 10$ equation (1a) gives $n_i = 0.62$, which we round off to the nearest integer, 1, giving one spheron (2 neutrons) in the core and 9 spherons (18 neutrons) in the mantle. The neutron configuration is thus found to be $1s^2$ for the core and $2s^2 1p^6 1d^{10}$ for the mantle. The magic number 20 is seen to correspond to completed shells for both the core (K shell) and the mantle (M shell).

The heavy bar in Figure 2 indicating completion of the K shell of neutrons in the core extends from $N = 14.4$ to $N = 26.8$. These limits correspond to 1.5 ± 1.0 neutrons in the core, 1.5 being the value for transition from $1s$ to $1s^2$, and ± 1 representing the uncertainty in the equation. The bars for other completed shells have been similarly drawn, and those for completed subsubshells have been drawn with only half this width (the uncertainty, however, is as great).

In the calculations for the inner core and innermost core the decreased volume of the inner spherons (shift to tritons and dineutrons) has been taken into account. The nature of the spherons is indicated by the chart (Fig. 2) and the Z/N ratio.

To within its reliability, Figure 2 applies to protons as well as to neutrons.

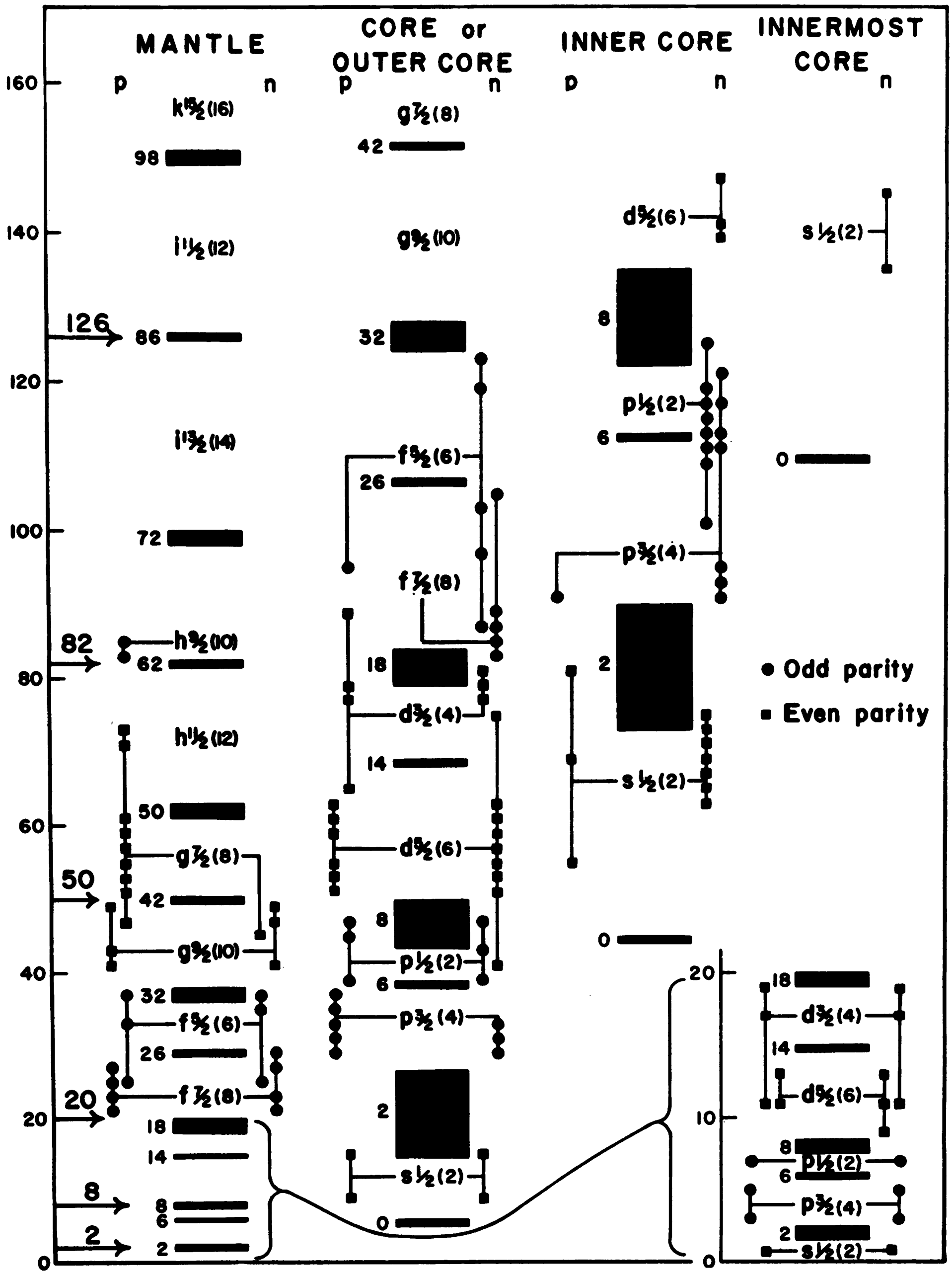

FIG. 2.—A diagram showing the ranges of values of the neutron number N in which successive subsubshells of the mantle, outer core, and inner core are occupied by neutrons, as calculated with use of the packing equation. Observed values of spin and parity of odd N and odd Z nuclei are indicated by circles and squares.

The sequence of levels shown in Figure 2 closely resembles the level diagram found by Mayer and Jensen by analysis of observed nuclear properties, with the help of the calculated level sequences for harmonic-oscillator and square-well potential functions.[9] Our packing equation provides a simple theoretical explanation of the sequence of levels. Moreover, it shows the regions of overlapping of the ranges for mantle and core subsubshells. When the spins are much different, these are the regions of existence of isomers.

Nuclear Spins and Parities.—The level sequence of Mayer and Jensen was largely derived from observed values of nuclear spins and parities. The extent to which these observations agree with the ranges given by the packing equation is shown in Figure 2. Most of the points lie within the ranges of N and Z for which a single unpaired nucleon would have the observed spin and parity (as indicated by the connecting lines). It has been pointed out by Mayer and Jensen that some of the apparent exceptions may indicate that three or five unpaired nucleons are coupled to a resultant spin; for example, a state with $J = 3/2$ and positive parity, as observed for both $Z = 11$ and $N = 11$, is an allowed state for the configuration $(d5/2)^3$.

The circles and dots in Figure 2 represent 251 odd-N or odd-Z nuclei for which spins and parities are given on the 1964 General Electric Chart of the Nuclides. The values for eleven nuclides, not accounted for by the level diagram, are not shown in the figure; they are ${}_{67}Ho^{161,\,165}(7/2-)$, ${}_{93}Np^{237,\,239}(5/2+)$, $Se_{41}{}^{75}(5/2+)$, $Gd_{89}{}^{153}(3/2+)$, $Dy_{95}{}^{161}(5/2+)$, $Dy_{99}{}^{165}$ and $Er_{99}{}^{167}(7/2+)$, and $U_{143}{}^{235}$ and $Pu_{143}{}^{237}$ $(7/2-)$.

The Structural Basis of the Magic Numbers.—Elsasser[10] in 1933 pointed out that certain numbers of neutrons or protons in an atomic nucleus confer increased stability on it. These numbers, called magic numbers, played an important part in the development of the shell model;[4, 5] it was found possible to associate them with configurations involving a spin-orbit subsubshell, but not with any reasonable combination of shells and subshells alone. The shell-model level sequence in its usual form,[11] however, leads to many numbers at which subsubshells are completed, and provides no explanation of the selection of a few of them (6 of 25 in the range 0–170) as magic numbers.

The packing diagram given in Figure 2 provides a simple structural basis of the magic numbers.[1]

The magic numbers[12] are 2, 8, 20, 50, 82, and 126. Reference to Figure 2 shows that they are distinguished from all other numbers by having a single layer that is a completed shell or a core consisting of a completed shell or shells. Thus 2 is the completed K shell, $1s^2$, and 8 the completed L shell, $1s^21p^6$; 20 consists of an M shell, $2s^21p^61d^{10}$, as mantle, and a K shell as core; 50 has L as core, 82 has M as outer core and K as inner core, and 126 has N as outer core and L as inner core. For 50, 82, and 126 the mantle, with the number of nucleons required by the packing equation to surround the core, consists of a completed shell plus a subsubshell. The magic-number configurations are given in Table 1.

TABLE 1

NUCLEON CONFIGURATIONS FOR THE MAGIC NUMBERS

Magic number	Mantle	Core or outer core	Inner core
2	$1s^2$		
8	$1s^21p^6$		
20	$2s^21p^61d^{10}$	$1s^2$	
50	$2s^22p^61d^{10}1f^{14}(1g\ 9/2)^{10}$	$1s^21p^6$	
82	$3s^22p^62d^{10}1f^{14}1g^{18}(1h\ 11/2)^{12}$	$2s^21p^61d^{10}$	$1s^2$
126	$3s^23p^62d^{10}2f^{14}1g^{18}1h^{22}(1i\ 13/2)^{14}$	$2s^22p^61d^{10}1f^{14}$	$1s^21p$

Summary.—The assumption that atomic nuclei consist of closely packed spherons (aggregates of neutrons and protons in localized 1*s* orbitals—mainly helions and tritions) in concentric layers leads to a simple derivation of a subsubshell occupancy diagram for nucleons and a simple explanation of magic numbers. Application of the close-packed-spheron model of the nucleus to other problems, including that of asymmetric fission, will be published later.[13]

* Fellow of the John Simon Guggenheim Memorial Foundation.

[1] The structural interpretation of the principal quantum number of nucleonic orbital wave functions and the structural basis provided by the close-packed-spheron theory for the neutron and proton magic numbers are discussed in notes submitted to *Phys. Rev. Letters* and *Nature* (L. Pauling, 1965).

[2] The name "helion" is used for the alpha particle: Pauling, L., *Nature*, **201**, 61 (1964).

[3] A review of this field is given by Goldhammer, P., *Rev. Mod. Phys.*, **35**, 40 (1963).

[4] Mayer, M. Goeppert, *Phys. Rev.*, **75**, 1969 (1949).

[5] Haxel, O., J. H. D. Jensen, and H. E. Suess, *Phys. Rev.*, **75**, 1766 (1949); *Z. Physik*, **128**, 295 (1950).

[6] Mayer, M. Goeppert, and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (New York and London: John Wiley and Sons, Inc., 1955).

[7] Hofstadter, R., *Ann. Rev. Nucl. Sci.*, **7**, 231 (1957).

[8] Bergman, G., J. L. T. Waugh, and L. Pauling, *Nature*, **169**, 1057 (1952); *Acta Cryst.*, **10**, 254 (1957).

[9] Ref. 6, p. 83.

[10] Elsasser, W. M., *J. Phys. Radium*, **4**, 549 (1933); **5**, 389, 635 (1934).

[11] Ref. 6, p. 58.

[12] The set of magic numbers is often considered to include 28. The amount of extra stability at Z or $N = 28$ is, however, much less than at 2, 8, 20, 50, 82, and 126, and I have chosen to exclude it.

[13] Pauling, L., *Science*, in press.

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*RED SHIFT DISCRETIZATION IN THE EPOCHAL COSMOLOGY**

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1. *Introduction.*—A. G. Wilson has found that the empirical relation

$$\frac{\bar{z}}{1+\bar{z}} = \frac{M(M+1)N(N+1)}{(137.5)^2}, \tag{1}$$

in which M and N are integers, can be used to determine the mean $\bar{z}$ of the spectral displacements or red shifts $\Delta\lambda/\lambda$ of clusters of galaxies. His results are shown in Table 1, which has been reproduced from his original paper.[1] The cluster designation and the common name of the cluster are given in the first and second columns of the table; also the quantity $\bar{u}_0$ in the table is the reciprocal of the left member of the relation (1) and the quantity P is given by the expression

$$P = 4.2765 - \log M(M+1) - \log N(N+1).$$

In view of the extraordinary accuracy with which the formula (1) is capable of repre-